Paper Specific Instructions

1. The examination is of 3 hours duration. There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, **A**, **B** and **C**. All sections are compulsory. Questions in each section are of different types.

2. **Section A** contains a total of 30 **Multiple Choice Questions (MCQ).** Each MCQ type question has four choices out of which only **one** choice is the correct answer. Questions Q.1 – Q.30 belong to this section and carry a total of 50 marks. Q.1 – Q.10 carry 1 mark each and Questions Q.11 – Q.30 carry 2 marks each.

3. **Section B** contains a total of 10 **Multiple Select Questions (MSQ).** Each MSQ type question is similar to MCQ but with a difference that there will be **more than one** choices that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.31 – Q.40 belong to this section and carry 2 marks each with a total of 20 marks.

4. **Section C** contains a total of 20 **Numerical Answer Type (NAT)** questions. For these NAT type questions, the answer is a real number which needs to be entered using the virtual keyboard on the monitor. No choices will be shown for these type of questions. Questions Q.41 – Q.60 belong to this section and carry a total of 30 marks. Q.41 – Q.50 carry 1 mark each and Questions Q.51 – Q.60 carry 2 marks each.

5. In all sections, questions not attempted will result in zero mark. In **Section A** (MCQ), wrong answer will result in **NEGATIVE** marks. For all 1-mark questions, 1/3 marks will be deducted for each wrong answer. For all 2-mark questions, 2/3 marks will be deducted for each wrong answer. In **Section B** (MSQ), there is **NO NEGATIVE** and **NO PARTIAL** marking provisions. There is **NO NEGATIVE** marking in **Section C** (NAT) as well.

6. Only Virtual Scientific Calculator is allowed. Charts, graph sheets, tables, cellular phone or other electronic gadgets are **NOT** allowed in the examination hall.

7. A Scribble Pad will be provided for rough work.

Special Instructions / Useful Data

$\mathbb{N}$ = The set of all natural numbers

$\mathbb{Z}$ = The set of all integers

$\mathbb{Z}_n = \{\overline{0}, \overline{1}, \dots, \overline{n-1}\}$, the group of integers modulo n, under addition modulo n, for $n \in \mathbb{N}$

$\mathbb{R}$ = The set of all real numbers

$\mathbb{R}^n =$ The $n-$dimensional Euclidean space

ln x = The natural logarithm of x (to the base e)

S_n = The symmetric group of all permutations on $\{1,2, \dots, n\}$

id = The identity element in S_n

(a_n) = The infinite sequence $a_1, a_2, a_3, \ . \ . \ .$

$f \circ g$ = The composition of f and g, defined by $(f \circ g)(x) = f(g(x))$

$f'(x)$ = The first derivative of f at the point x

$f''(x)$ = The second derivative of f at the point x

span S = The linear span of the subset S of a vector space

$P_n(\mathbb{R})$ = The real vector space of real polynomials of degree less than or equal to n, together with the zero polynomial. These polynomials can be regarded as functions from $\mathbb{R}$ to $\mathbb{R}$

$\ker(T)$ = The kernel of the linear transformation T

$M = (m_{ij})$ = Matrix of appropriate order with the entry/element in the i^{th} row and j^{th} column denoted by m_{ij}, $m_{ij} \in \mathbb{R}$

$\gcd(m, n)$ = The greatest common divisor of the natural numbers m and n

$\det(M)$ = The determinant of the matrix M

$\frac{\partial f}{\partial x}$ = The partial derivative of f with respect to x

$\frac{\partial f}{\partial y}$ = The partial derivative of f with respect to y

Section A: Q.1 – Q.10 Carry ONE mark each.

Q.1 The sum of the infinite series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\pi^{2n+1}}{2^{2n+1}(2n)!}$$

is equal to

(A) $-\pi$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{2}$

(D) $-\frac{\pi}{4}$

Q.2 For which one of the following choices of $N(x,y)$, is the equation

$$(e^x \sin y - 2y \sin x)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$$

an exact differential equation?

(A) $N(x,y) = e^x \sin y + 2\cos x$

(B) $N(x,\mathrm{y}) = e^x \cos y + 2\cos x$

(C) $N(x,y) = e^x \cos y + 2\sin x$

(D) $N(x,y) = e^x \sin y + 2\sin x$

Q.3 Let $f\,,\, g : \mathbb{R} \to \mathbb{R}$ be two functions defined by

$$f(x) = \begin{cases} x\,|x|\left|\sin\dfrac{1}{x}\right| & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} x^2\sin\dfrac{1}{x} + x\cos\dfrac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0\,. \end{cases}$$

Then, which one of the following is TRUE?

(A) f is differentiable at $x = 0$, and g is NOT differentiable at $x = 0$

(B) f is NOT differentiable at $x = 0$, and g is differentiable at $x = 0$

(C) f is differentiable at $x = 0$, and g is differentiable at $x = 0$

(D) f is NOT differentiable at $x = 0$, and g is NOT differentiable at $x = 0$

Q.4 Let $f,\ g : \mathbb{R} \to \mathbb{R}$ be two functions defined by

$$f(x) = \begin{cases} |x|^{1/8} \left|\sin \frac{1}{x}\right| \cos x & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} e^x \cos \frac{1}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}.$$

Then, which one of the following is TRUE?

(A) f is continuous at $x = 0$, and g is NOT continuous at $x = 0$

(B) f is NOT continuous at $x = 0$, and g is continuous at $x = 0$

(C) f is continuous at $x = 0$, and g is continuous at $x = 0$

(D) f is NOT continuous at $x = 0$, and g is NOT continuous at $x = 0$

Q.5 Which one of the following is the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 2e^{4x} \ ?$$

(A) $\alpha_1 e^{4x} + \alpha_2 x e^{4x} + x^2 e^{4x}$, where $\alpha_1, \alpha_2 \in \mathbb{R}$

(B) $\alpha_1 e^{4x} + \alpha_2 x e^{4x} + 2x^2 e^{4x}$, where $\alpha_1, \alpha_2 \in \mathbb{R}$

(C) $\alpha_1 e^{-4x} + \alpha_2 e^{4x} + 2x^2 e^{4x}$, where $\alpha_1, \alpha_2 \in \mathbb{R}$

(D) $\alpha_1 x e^{-4x} + \alpha_2 x^2 e^{-4x} + x^2 e^{4x}$, where $\alpha_1, \alpha_2 \in \mathbb{R}$

Q.6 Define $T: \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(x, y, z) = (x + z,\ 2x + 3y + 5z,\ 2y + 2z), \text{ for all } (x, y, z) \in \mathbb{R}^3 .$$

Then, which one of the following is TRUE?

(A) T is one-one and T is NOT onto

(B) T is NOT one-one and T is onto

(C) T is one-one and T is onto

(D) T is NOT one-one and T is NOT onto

Q.7 Let $M = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & x \end{pmatrix}$ for some real number x.

If 0 is an eigenvalue of M, then $(M^4 + M)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is equal to

(A) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

(B) $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$

(C) $\begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix}$

(D) $\begin{pmatrix} 17 \\ 0 \\ 17 \end{pmatrix}$

Q.8 Let $\mathrm{T} : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $\mathrm{T}\big(p(x)\big) = p(x+1)$, for all $p(x) \in P_2(\mathbb{R})$. If M is the matrix representation of T with respect to the ordered basis $\{1, x, x^2\}$ of $P_2(\mathbb{R})$, then which one of the following is TRUE?

(A) The determinant of M is 2

(B) The rank of M is 2

(C) 1 is the only eigenvalue of M

(D) The nullity of M is 2

Q.9 Let G be a finite abelian group of order 10. Let x_0 be an element of order 2 in G.

If $X = \{x \in G : x^3 = x_0\}$, then which one of the following is TRUE?

(A) X has exactly one element

(B) X has exactly two elements

(C) X has exactly three elements

(D) X is an empty set

Q.10 The value of

$$\int_0^1 \left(\int_{\sqrt{y}}^1 3e^{x^3}\, dx \right) dy$$

is equal to

(A) $e - 1$

(B) $\frac{e-1}{2}$

(C) $\sqrt{e} - 1$

(D) $\frac{\sqrt{e}-1}{2}$

Section A: Q.11 – Q.30 Carry TWO marks each.

Q.11 Let $\mathcal{C}$ denote the family of curves described by $yx^2 = \lambda$, for $\lambda \in (0, \infty)$ and lying in the first quadrant of the xy plane. Let $\mathcal{O}$ denote the family of orthogonal trajectories of $\mathcal{C}$.

Which one of the following curves is a member of $\mathcal{O}$, and passes through the point $(2, 1)$?

(A) $y = \frac{x^2}{4}$, $x > 0$, $y > 0$

(B) $x^2 - 2y^2 = 2$, $x > 0$, $y > 0$

(C) $x - y = 1$, $x > 0$, $y > 0$

(D) $2x - y^2 = 3$, $x > 0$, $y > 0$

Q.12 Let $\varphi : (0, \infty) \to \mathbb{R}$ be the solution of the differential equation

$$x\frac{dy}{dx} = (\ln y - \ln x)y,$$

satisfying $\varphi(1) = e^2$. Then, the value of $\varphi(2)$ is equal to

(A) e^2

(B) $2e^3$

(C) $3e^2$

(D) $6e^3$

Q.13 Let $X = \{x \in S_4 : x^3 = id\}$ and $Y = \{x \in S_4 : x^2 \neq id\}$.

If m and n denote the number of elements in X and Y, respectively, then which one of the following is TRUE?

(A) m is even and n is even

(B) m is odd and n is even

(C) m is even and n is odd

(D) m is odd and n is odd

Q.14 Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the solution of the differential equation

$$\frac{dy}{dx} = (y-1)\,(y-3)\,,$$

satisfying $\varphi(0) = 2$. Then, which one of the following is TRUE?

(A) $\lim_{x\to\infty} \varphi(x) = 0$

(B) $\lim_{x\to\ln\sqrt{2}} \varphi(x) = 1$

(C) $\lim_{x\to-\infty} \varphi(x) = 3$

(D) $\lim_{x\to\ln\frac{1}{\sqrt{2}}} \varphi(x) = 6$

Q.15 Let $M = \begin{pmatrix} 6 & 2 & -6 & 8 \\ 5 & 3 & -9 & 8 \\ 3 & 1 & -2 & 4 \end{pmatrix}$. Consider the system S of linear equations given by

$$\begin{gathered} 6x_1 + 2x_2 - 6x_3 + 8x_4 = 8 \\ 5x_1 + 3x_2 - 9x_3 + 8x_4 = 16 \\ 3x_1 + x_2 - 2x_3 + 4x_4 = 32 \end{gathered}$$

where x_1, x_2, x_3, x_4 are unknowns.

Then, which one of the following is TRUE?

(A) The rank of M is 3, and the system S has a solution

(B) The rank of M is 3, and the system S does NOT have a solution

(C) The rank of M is 2, and the system S has a solution

(D) The rank of M is 2, and the system S does NOT have a solution

Q.16 Let $M = \begin{pmatrix} -2 & 0 & 0 \\ 3 & 2 & 3 \\ 4 & -1 & x \end{pmatrix}$ for some real number x. Suppose that -2 and 3 are eigenvalues of M. If $M^3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 125 \\ 125 \end{pmatrix}$, then which one of the following is TRUE?

(A) $x = 5$, and the matrix $M^2 + M$ is invertible

(B) $x \neq 5$, and the matrix $M^2 + M$ is invertible

(C) $x = 5$, and the matrix $M^2 + M$ is NOT invertible

(D) $x \neq 5$, and the matrix $M^2 + M$ is NOT invertible

Q.17 Let $f(x) = 10x^2 + e^x - \sin(2x) - \cos x,\ x \in \mathbb{R}$. The number of points at which the function f has a local minimum is

(A) 0

(B) 1

(C) 2

(D) greater than or equal to 3

Q.18 For $n \in \mathbb{N}$, define x_n and y_n by

$$x_n = (-1)^n \cos\frac{1}{n} \quad \text{and} \quad y_n = \sum_{k=1}^{n} \frac{1}{n+k}.$$

Then, which one of the following is TRUE?

(A) $\sum_{n=1}^{\infty} x_n$ converges, and $\sum_{n=1}^{\infty} y_n$ does NOT converge

(B) $\sum_{n=1}^{\infty} x_n$ does NOT converge, and $\sum_{n=1}^{\infty} y_n$ converges

(C) $\sum_{n=1}^{\infty} x_n$ converges, and $\sum_{n=1}^{\infty} y_n$ converges

(D) $\sum_{n=1}^{\infty} x_n$ does NOT converge, and $\sum_{n=1}^{\infty} y_n$ does NOT converge

Q.19 Let $x_1 = \frac{5}{2}$. For $n \in \mathbb{N}$, define

$$x_{n+1} = \frac{1}{5}(x_n^2 + 6).$$

Then, which one of the following is TRUE?

(A) (x_n) is an increasing sequence, and (x_n) is NOT a bounded sequence

(B) (x_n) is NOT an increasing sequence, and (x_n) is NOT a bounded sequence

(C) (x_n) is NOT a decreasing sequence, and (x_n) is a bounded sequence

(D) (x_n) is a decreasing sequence, and (x_n) is a bounded sequence

Q.20 Let $x_1 = 2$ and $x_{n+1} = 2 + \frac{1}{2x_n}$ for all $n \in \mathbb{N}$.

Then, which one of the following is TRUE?

(A) $x_{n+1} \geq \frac{4}{x_n}$ for all $n \in \mathbb{N}$, and (x_n) is a Cauchy sequence

(B) $x_{n+1} < \frac{4}{x_n}$ for some $n \in \mathbb{N}$, and (x_n) is a Cauchy sequence

(C) $x_{n+1} \geq \frac{4}{x_n}$ for all $n \in \mathbb{N}$, and (x_n) is NOT a Cauchy sequence

(D) $x_{n+1} < \frac{4}{x_n}$ for some $n \in \mathbb{N}$, and (x_n) is NOT a Cauchy sequence

Q.21 For $n \in \mathbb{N}$, define x_n and y_n by

$$x_n = (-1)^n \frac{3^n}{n^3} \text{ and } y_n = \left(4^n + (-1)^n 3^n\right)^{1/n}.$$

Then, which one of the following is TRUE?

(A) (x_n) has a convergent subsequence, and NO subsequence of (y_n) is convergent

(B) NO subsequence of (x_n) is convergent, and (y_n) has a convergent subsequence

(C) (x_n) has a convergent subsequence, and (y_n) has a convergent subsequence

(D) NO subsequence of (x_n) is convergent, and NO subsequence of (y_n) is convergent

Q.22 Let $M = (m_{ij})$ be a 3×3 real, invertible matrix and $\sigma \in S_3$ be the permutation defined by $\sigma(1) = 2$, $\sigma(2) = 3$ and $\sigma(3) = 1$. The matrix $M_\sigma = (n_{ij})$ is defined by $n_{ij} = m_{i\sigma(j)}$ for all $i, j \in \{1, 2, 3\}$.

Then, which one of the following is TRUE?

(A) $\det(M) = \det(M_\sigma)$, and nullity of the matrix $M - M_\sigma$ is 0

(B) $\det(M) = -\det(M_\sigma)$, and nullity of the matrix $M - M_\sigma$ is 1

(C) $\det(M) = \det(M_\sigma)$, and nullity of the matrix $M - M_\sigma$ is 1

(D) $\det(M) = -\det(M_\sigma)$, and nullity of the matrix $M - M_\sigma$ is 0

Q.23 Let $\mathbb{R}/\mathbb{Z}$ denote the quotient group, where $\mathbb{Z}$ is considered as a subgroup of the additive group of real numbers $\mathbb{R}$.

Let m denote the number of injective (one-one) group homomorphisms from $\mathbb{Z}_3$ to $\mathbb{R}/\mathbb{Z}$ and n denote the number of group homomorphisms from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{Z}_3$.

Then, which one of the following is TRUE?

(A) $m = 2$ and $n = 1$

(B) $m = 3$ and $n = 3$

(C) $m = 2$ and $n = 3$

(D) $m = 1$ and $n = 1$

Q.24 Let f_1, f_2, f_3 be nonzero linear transformations from $\mathbb{R}^4$ to $\mathbb{R}$ and

$$\ker(f_1) \subseteq \ker(f_2) \cap \ker(f_3)\,.$$

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(v) = \big(f_1(v), f_2(v), f_3(v)\big)\,, \qquad \text{for all } v \in \mathbb{R}^4\,.$$

Then, the nullity of T is equal to

(A) 1

(B) 2

(C) 3

(D) 4

Q.25 Let $x_1 = 1$. For $n \in \mathbb{N}$, define

$$x_{n+1} = \left(\frac{1}{2} + \frac{\sin^2 n}{n}\right) x_n .$$

Then, which one of the following is TRUE?

(A) $\sum_{n=1}^{\infty} x_n$ converges

(B) $\sum_{n=1}^{\infty} x_n$ does NOT converge

(C) $\sum_{n=1}^{\infty} x_n^2$ does NOT converge

(D) $\sum_{n=1}^{\infty} x_n x_{n+1}$ does NOT converge

Q.26 Let $x_1 > 0$. For $n \in \mathbb{N}$, define $x_{n+1} = x_n + 4$. If

$$\lim_{n\to\infty}\left(\frac{1}{x_2x_3}+\frac{1}{x_3x_4}+\cdots+\frac{1}{x_{n+1}x_{n+2}}\right)=\frac{1}{24},$$

then the value of x_1 is equal to

(A) 1

(B) 2

(C) 3

(D) 8

Q.27 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = e^y (x^2 + y^2) \quad \text{for all } (x, y) \in \mathbb{R}^2 .$$

Then, which one of the following is TRUE?

(A) The number of points at which f has a local minimum is 2

(B) The number of points at which f has a local maximum is 2

(C) The number of points at which f has a local minimum is 1

(D) The number of points at which f has a local maximum is 1

Q.28 Let Ω be the bounded region in $\mathbb{R}^3$ lying in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$), and bounded by the surfaces $z = x^2 + y^2$, $z = 4$, $x = 0$ and $y = 0$.

Then, the volume of Ω is equal to

(A) π

(B) 2π

(C) 3π

(D) 4π

Q.29 Let $\varphi : [0, \infty) \to \mathbb{R}$ be the continuous function satisfying

$$\varphi(x) = \left(\int_0^x \varphi(t) dt \right) + \sin x \ , \quad \text{for all } x \in [0, \infty) \, .$$

Then, the value of $\lim_{x \to \pi/2} (2\varphi(x) - e^x)$ is equal to

(A) 1

(B) 2

(C) 3

(D) 4

Q.30 The number of elements in the set

$$\{x \in \mathbb{R} : 8x^2 + x^4 + x^8 = \cos x\}$$

is equal to

(A) 0

(B) 1

(C) 2

(D) greater than or equal to 3

Section B: Q.31 – Q.40 Carry TWO marks each.

Q.31 Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2 + y^5}{x^2 + y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Then, which of the following is/are TRUE?

(A) The iterated limits $\lim\limits_{x \to 0}\left(\lim\limits_{y \to 0} f(x,y)\right)$ and $\lim\limits_{y \to 0}\left(\lim\limits_{x \to 0} f(x,y)\right)$ exist

(B) Exactly one of the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exists at $(0,0)$

(C) Both the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$

(D) f is NOT differentiable at $(0,0)$

Q.32 If $M, N, \mu, w : \mathbb{R}^2 \to \mathbb{R}$ are differentiable functions with continuous partial derivatives, satisfying

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = dw$$

then which of the following is/are TRUE?

(A) μw is an integrating factor for $M(x,y)dx + N(x,y)dy = 0$

(B) μw^2 is an integrating factor for $M(x,y)dx + N(x,y)dy = 0$

(C) $w(x,y) = w(0,0) + \int_0^x (\mu M)(s,0)ds + \int_0^y (\mu N)(x,t)dt$, for all $(x,y) \in \mathbb{R}^2$

(D) $w(x,y) = w(0,0) + \int_0^x (\mu M)(s,y)ds + \int_0^x (\mu N)(0,t)dt$, for all $(x,y) \in \mathbb{R}^2$

Q.33 Let $\varphi : (-1, \infty) \to (0, \infty)$ be the solution of the differential equation

$$\frac{dy}{dx} - 2\,ye^{x} = 2e^{x}\sqrt{y}\ ,$$

satisfying $\varphi(0) = 1$.

Then, which of the following is/are TRUE?

(A) φ is an unbounded function

(B) $\lim_{x \to \ln 2} \varphi(x) = (2e - 1)^2$

(C) $\lim_{x \to \ln 2} \varphi(x) = \sqrt{2e - 1}$

(D) φ is a strictly increasing function on the interval $(0, \infty)$

Q.34 Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{(x^2 + \sin x)y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Then, which of the following is/are TRUE?

(A) $\lim_{(x,y)\to(0,0)} f(x,y)$ exists and $\lim_{(x,y)\to(0,0)} f(x,y) = 1$

(B) $\lim_{(x,y)\to(0,0)} f(x,y)$ exists and $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

(C) f is differentiable at $(0,0)$

(D) f is NOT differentiable at $(0,0)$

Q.35 Let

$u_1 = (1,0,0,-1), u_2 = (0,2,0,-1),\ \ u_3 = (0,0,1,-1)$ and $u_4 = (0,0,0,1)$

be elements in the real vector space $\mathbb{R}^4$.

Then, which of the following is/are TRUE?

(A) $\{u_1,\ u_2,\ u_3,\ u_4\}$ is a linearly independent set in $\mathbb{R}^4$

(B) $\{u_1 - u_2, u_2 - u_3,\ u_3 - u_4, u_4 - u_1\}$ is NOT a linearly independent set in $\mathbb{R}^4$

(C) $\{u_1,\ -u_2,\ \ u_3, -u_4\}$ is NOT a linearly independent set in $\mathbb{R}^4$

(D) $\{u_1 + u_2,\ u_2 +\ u_3,\ u_3 + u_4,\ u_4 + u_1\}$ is a linearly independent set in $\mathbb{R}^4$

Q.36 For $n \in \mathbb{N}$, let

$$x_n = \sum_{k=1}^{n} \frac{k}{n^2 + k}.$$

Then, which of the following is/are TRUE?

(A) The sequence (x_n) converges

(B) The series $\sum_{n=1}^{\infty} x_n$ converges

(C) The series $\sum_{n=1}^{\infty} x_n$ does NOT converge

(D) The series $\sum_{n=1}^{\infty} x_n^n$ converges

Q.37 Let $f: \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that

$$f(0) = 0, f'(0) = 2 \text{ and } f(1) = -3\,.$$

Then, which of the following is/are TRUE?

(A) $|f'(x)| \leq 2$ for all $x \in [0, 1]$

(B) $|f'(x_1)| > 2$ for some $x_1 \in [0, 1]$

(C) $|f''(x)| < 10$ for all $x \in [0, 1]$

(D) $|f''(x_2)| \geq 10$ for some $x_2 \in [0, 1]$

Q.38 Let $f: \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that

$$f(0) = 4, f(1) = -2, f(2) = 8 \text{ and } f(3) = 2\,.$$

Then, which of the following is/are TRUE?

(A) $|f'(x)| < 5$ for all $x \in [0,1]$

(B) $|f'(x_1)| \geq 5$ for some $x_1 \in [0,1]$

(C) $f'(x_2) = 0$ for some $x_2 \in [0,3]$

(D) $f''(x_3) = 0$ for some $x_3 \in [0,3]$

Q.39 For $n \in \mathbb{N}$, consider the set $U(n) = \{\bar{x} \in \mathbb{Z}_n : \gcd(x, n) = 1\}$ as a group under multiplication modulo n.

Then, which of the following is/are TRUE?

(A) $U(8)$ is a cyclic group

(B) $U(5)$ is a cyclic group

(C) $U(12)$ is a cyclic group

(D) $U(9)$ is a cyclic group

Q.40 Consider the following subspaces of the real vector space $\mathbb{R}^3$:

$V_1 = \text{span}\,\{(1,2,3),(1,1,0)\}$,

$V_2 = \text{span}\,\{(1,-1,0)\}$,

$V_3 = \text{span}\,\{(1,1,1)\}$,

$V_4 = \text{span}\,\{(1,3,6)\}$ and

$V_5 = \text{span}\,\{(1,0,-3)\}$.

Then, which of the following is/are TRUE?

(A) $V_1 \cup V_2$ is a subspace of $\mathbb{R}^3$

(B) $V_1 \cup V_3$ is a subspace of $\mathbb{R}^3$

(C) $V_1 \cup V_4$ is a subspace of $\mathbb{R}^3$

(D) $V_1 \cup V_5$ is a subspace of $\mathbb{R}^3$

Section C: Q.41 – Q.50 Carry ONE mark each.

Q.41 The radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{\left(x+\frac{1}{4}\right)^n}{(-2)^n n^2}$$

about $x = -\frac{1}{4}$, is equal to ______________ *(rounded off to two decimal places).*

Q.42 The value of

$$\lim_{n\to\infty} 8n \left(e^{\left(\frac{1}{2n}\right)} - 1\right)\left(\sin\frac{1}{2n} + \left|\cos\frac{1}{2n}\right|\right)$$

is equal to ________________ *(rounded off to two decimal places).*

Q.43 Let α be the real number such that

$$\lim_{x\to 0} \frac{(1-\cos x)(2^{2+x}-4)}{x^3} = \alpha \ln 2 \; .$$

Then, the value of α is equal to ____________ *(rounded off to two decimal places).*

Q.44 Let $\varphi: \mathbb{R} \to \mathbb{R}$ be the solution of the differential equation

$$4\frac{d^2y}{dx^2} + 16\,\frac{dy}{dx} + 25y = 0$$

satisfying $\varphi(0) = 1$ and $\varphi'(0) = -\frac{1}{2}$.

Then, the value of $\lim_{x\to\pi/6} e^{2x}\varphi(x)$ is equal to _____________ *(rounded off to two decimal places).*

Q.45 Let S be the surface area of the portion of the plane $z = x + y + 3$, which lies inside the cylinder $x^2 + y^2 = 1$.

Then, the value of $\left(\frac{S}{\pi}\right)^2$ is equal to ____________ *(rounded off to two decimal places).*

Q.46 Consider the following subspaces of $\mathbb{R}^4$:

$$V_1 = \{(x, y, z, w) \in \mathbb{R}^4 : x + y + 2w = 0\},$$

$$V_2 = \{(x, y, z, w) \in \mathbb{R}^4 : 2y + z + w = 0\},$$

$$V_3 = \{(x, y, z, w) \in \mathbb{R}^4 : x + 3y + z + 3w = 0\}.$$

Then, the dimension of the subspace $V_1 \cap V_2 \cap V_3$ is equal to __________.

Q.47 Consider the real vector space $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation such that

$$T(1,1,1) = 0, \qquad T(1,-1,1) = 0 \text{ and } T(0,0,1) = 16.$$

Then, the value of $T\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}\right)$ is equal to ____________________ *(rounded off to two decimal places).*

Q.48 Let T denote the triangle in the xy plane bounded by the x axis and the lines $y = x$ and $x = 1$. The value of the double integral (over T)

$$\iint_T (5-y)dxdy$$

is equal to __________ *(rounded off to two decimal places).*

Q.49 Let $T, S : P_4(\mathbb{R}) \to P_4(\mathbb{R})$ be the linear transformations defined by

$$T\big(p(x)\big) = xp'(x) \text{ and } S\big(p(x)\big) = (x+1)p'(x)$$

for all $p(x) \in P_4(\mathbb{R})$.

Then, the nullity of the composition $S \circ T$ is ______________.

Q.50 Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{(x^2-y^2)xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Then, the value of $\frac{\partial f}{\partial y}(1,0) - \frac{\partial f}{\partial x}(0,2)$ is equal to ______________ *(rounded off to two decimal places).*

Section C: Q.51 – Q.60 Carry TWO marks each.

Q.51 Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$\int_0^{\pi/4} \left(\sin(x)\, f(x) + \cos(x) \int_0^x f(t)dt \right) dx = \sqrt{2}\,.$$

Then, the value of $\int_0^{\pi/4} f(x)\, dx$ is equal to ________________ *(rounded off to two decimal places).*

Q.52 Let $\sigma \in S_4$ be the permutation defined by $\sigma(1) = 2,\ \sigma(2) = 3,\ \sigma(3) = 1$ and $\sigma(4) = 4$. The number of elements in the set

$$\{\tau \in S_4 : \tau\sigma\tau^{-1} = \sigma\}$$

is equal to _____________.

Q.53 Let $f(x) = 2x - \sin x$, for all $x \in \mathbb{R}$. Let $k \in \mathbb{N}$ be such that

$$\lim_{x \to 0} \left(\frac{1}{x} \sum_{i=1}^{k} i^2 f\left(\frac{x}{i}\right) \right) = 45.$$

Then, the value of k is equal to ______________.

Q.54 The value of the infinite series

$$\sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^{2(n-1)}$$

is equal to _______________ *(rounded off to two decimal places)*

Q.55 Let $\varphi: (0, \infty) \rightarrow \mathbb{R}$ be the solution of the differential equation

$$x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 6x \ln x$$

satisfying $\varphi(1) = -3$ and $\varphi(e) = 0$.

Then, the value of $|\varphi'(1)|$ is equal to ______________ *(rounded off to two decimal places).*

Q.56 Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be the solution of the differential equation

$$\frac{dy}{dx} + 2xy = 2 + 4x^2$$

satisfying $\varphi(0) = 0$.

Then, the value of $\varphi(2)$ is equal to ________________ *(rounded off to two decimal places).*

Q.57 Let Ω be the solid bounded by the planes $z = 0$, $y = 0$, $x = \frac{1}{2}$, $2y = x$ and $2x + y + z = 4$.

If V is the volume of Ω, then the value of $64\,V$ is equal to ____________ *(rounded off to two decimal places).*

Q.58 Let the subspace H of $P_3(\mathbb{R})$ be defined as

$$H = \{p(x) \in P_3(\mathbb{R}) : \; xp'(x) = 3p(x)\}.$$

Then, the dimension of H is equal to _________________.

Q.59 Let G be an abelian group of order 35. Let m denote the number of elements of order 5 in G, and let n denote the number of elements of order 7 in G.

Then, the value of $m + n$ is equal to ______________.

Q.60 The number of surjective (onto) group homomorphisms from S_4 to $\mathbb{Z}_6$ is equal to ________________.